BENHA UNIVERSITY
FACULTY OF ENGINEERING AT SHOUBRA

Post-Graduate
ECE-601
Active Circuits

Lecture #2
Scattering Matrices

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Agenda

- Introduction
- Microwave Network Analysis
- Scattering Matrix Representations
- Signal Flow Graphs

INTRODUCTION

Introduction

- Circuits operating at low frequencies can be treated as an interconnection of lumped passive or active components with unique voltages and currents defined at any point in the circuit.
- In this situation the circuit dimensions are small enough such that there is negligible phase delay from one point in the circuit to another.
- In addition, the fields can be considered as TEM fields supported by two or more conductors.
- This leads to a quasi-static type of solution to Maxwell's equations and to the well-known Kirchhoff voltage and current laws and impedance concepts of circuit theory .
- In general, the network analyzing techniques cannot be directly applied to microwave circuits, but the basic circuit and network concepts can be extended to handle many microwave analysis and design problems of practical interest.

MICROWAVE NETWORK ANALYSIS

Famous Matrices for Network Analysis

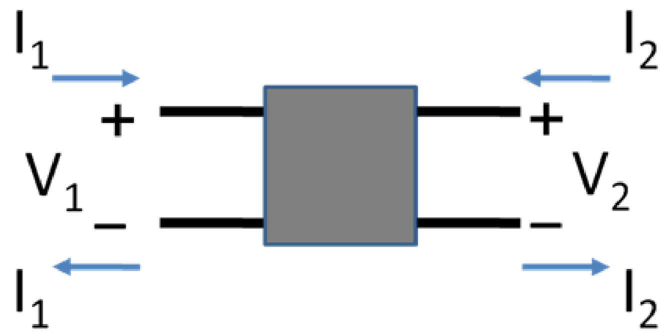
- Impedance (Z) Matrix
- Admittance (Y) Matrix
- Scattering (S) Matrix (*focus of the lecture*)
- Transmission (ABCD) Matrix

- Hybrid (h)
- Inverse hybrid (g)
- Scattering Transfer (T)

Matrices for 2-Port Network

A two-port network (a kind of four-terminal network or quadripole):

is an electrical network (circuit) or device with two pairs of terminals to connect to external circuits.



General Properties:

Reciprocal networks: A network is said to be reciprocal if the voltage appearing at port 2 due to a current applied at port 1 is the same as the voltage appearing at port 1 when the same current is applied to port 2.

Symmetrical networks: A network is symmetrical if its input impedance is equal to its output impedance.

Lossless network: A lossless network is one which contains no resistors or other dissipative elements.

Matrices Overview

- Z-Matrix

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

where

$$z_{11} \stackrel{\text{def}}{=} \left.\frac{V_1}{I_1}\right|_{I_2=0} \qquad z_{12} \stackrel{\text{def}}{=} \left.\frac{V_1}{I_2}\right|_{I_1=0}$$

$$z_{21} \stackrel{\text{def}}{=} \left.\frac{V_2}{I_1}\right|_{I_2=0} \qquad z_{22} \stackrel{\text{def}}{=} \left.\frac{V_2}{I_2}\right|_{I_1=0}$$

- Y-Matrix

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

where

$$y_{11} \stackrel{\text{def}}{=} \left.\frac{I_1}{V_1}\right|_{V_2=0} \qquad y_{12} \stackrel{\text{def}}{=} \left.\frac{I_1}{V_2}\right|_{V_1=0}$$

$$y_{21} \stackrel{\text{def}}{=} \left.\frac{I_2}{V_1}\right|_{V_2=0} \qquad y_{22} \stackrel{\text{def}}{=} \left.\frac{I_2}{V_2}\right|_{V_1=0}$$

- ABCD-Matrix

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix}$$

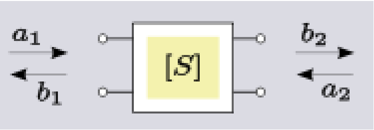
$$\begin{bmatrix} V_2 \\ I_2' \end{bmatrix} = \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} \begin{bmatrix} V_1 \\ I_1 \end{bmatrix}$$

where

$$A' \stackrel{\text{def}}{=} \left.\frac{V_2}{V_1}\right|_{I_1=0} \qquad B' \stackrel{\text{def}}{=} \left.\frac{V_2}{I_1}\right|_{V_1=0}$$

$$C' \stackrel{\text{def}}{=} -\left.\frac{I_2}{V_1}\right|_{I_1=0} \qquad D' \stackrel{\text{def}}{=} -\left.\frac{I_2}{I_1}\right|_{V_1=0}$$

- S-Matrix



$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

Z&Y Matrices General view

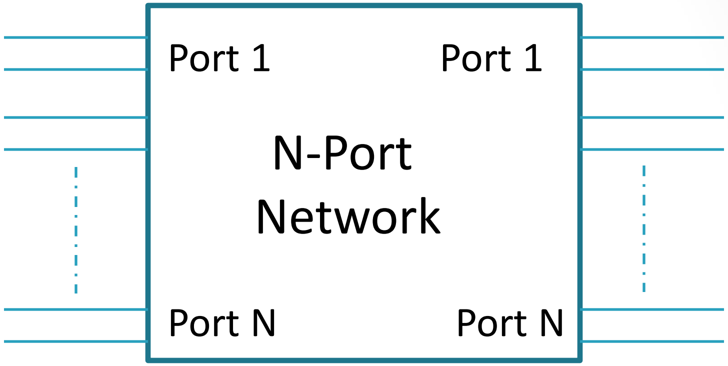
- Impedance

characteristic impedance of the medium $\eta = \sqrt{\frac{\mu}{\varepsilon}}$

wave impedance of the particular mode of wave $Z_w = \frac{E_t^+}{H_t^+}$

characteristic impedance of the line $Z_o = \frac{V^+}{I^+}$

input impedance at a port of circuit $Z_{in}(z) = \frac{V(z)}{I(z)}$



- Z-Matrix

$$[V]=[Z]\ [I]\ ,\ \begin{bmatrix} V_1 \\ V_2 \\ \bullet \\ \bullet \\ V_N \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} & \bullet & \bullet & Z_{1N} \\ Z_{21} & \bullet & \bullet & \bullet & Z_{2N} \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ Z_{N1} & Z_{N2} & \bullet & \bullet & Z_{NN} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ \bullet \\ \bullet \\ I_N \end{bmatrix},\ Z_{ij} = \left.\frac{V_i}{I_j}\right|_{I_k=0,k\neq j} = \left.\frac{response_i}{source_j}\right|_{I_k=0,k\neq j}$$

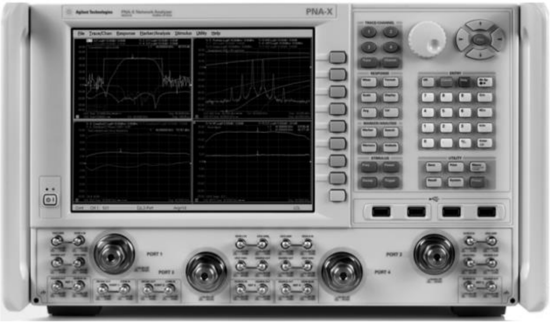
- Y-Matrix

$$[I]=[Y]\ [V],\begin{bmatrix} I_1 \\ I_2 \\ \bullet \\ \bullet \\ I_N \end{bmatrix} = \begin{bmatrix} Y_{11} & Y_{12} & \bullet & \bullet & Y_{1N} \\ Y_{21} & \bullet & \bullet & \bullet & Y_{2N} \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ Y_{N1} & Y_{N2} & \bullet & \bullet & Y_{NN} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ \bullet \\ \bullet \\ V_N \end{bmatrix},\ Y_{ij} = \left.\frac{I_i}{V_j}\right|_{V_k=0,k\neq j} = \left.\frac{response_i}{source_j}\right|_{V_k=0,k\neq j}$$

SCATTERING MATRIX REPRESENTATIONS

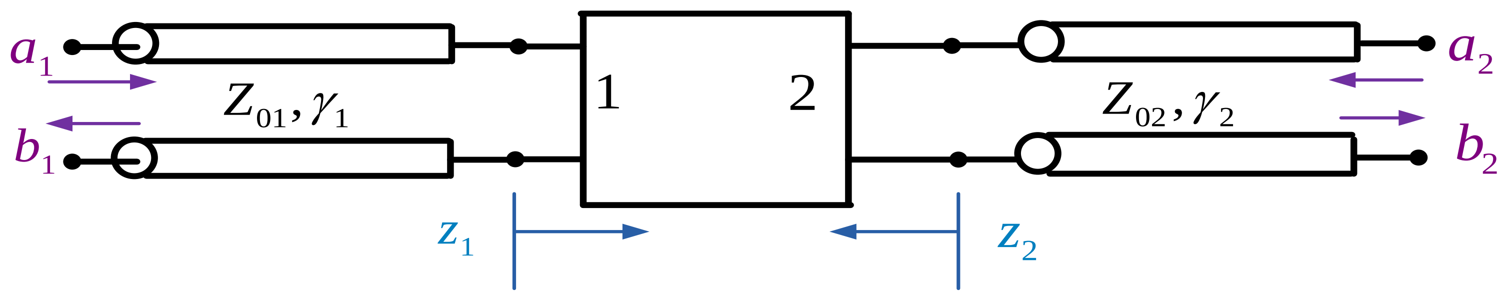
Motivation

- At high frequencies, Z, Y, h & ABCD parameters are difficult (if not impossible) to measure.
  - V and I are not uniquely defined
  - Even if defined, V and I are extremely difficult to measure (particularly I).
  - Required open and short-circuit conditions are often difficult to achieve.
  - In other words, direct measurements can't be done since all are EM waves at high frequencies
- Scattering (S) parameters are often the best representation for multi-port networks at high frequency.
- We can directly measure reflected, transmitted and incident wave with a network analyzer.
- Once the parameters are known they can be converted to any other matrix parameters.



Scattering Parameters

S-parameters are defined
assuming transmission lines are connected to each port.



On each transmission line:

$$V_i(z_i) = V_{i0}^{+} e^{-\gamma_i z_i} + V_{i0}^{-} e^{+\gamma_i z_i} = V_i^{+}(z_i) + V_i^{-}(z_i)$$

$$I_i(z_i) = \frac{V_i^{+}(z_i)}{Z_{0i}} - \frac{V_i^{-}(z_i)}{Z_{0i}} \qquad i = 1, 2$$

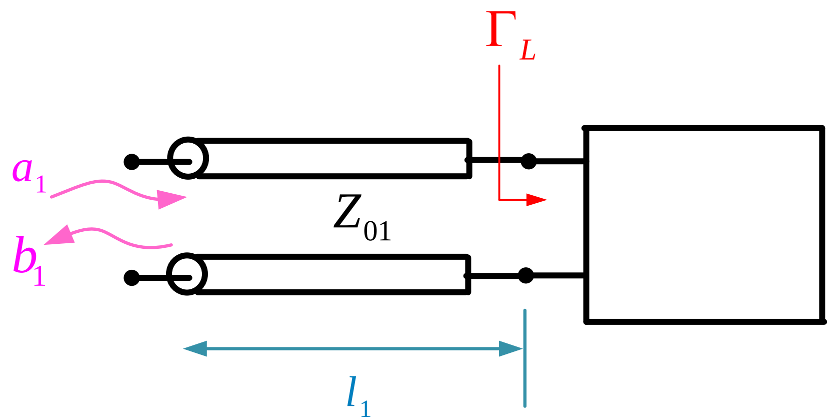
Incoming wave function $\equiv a_i(z_i) \equiv V_i^{+}(z_i) \big/ \sqrt{Z_{0i}}$

Outgoing wave function $\equiv b_i(z_i) \equiv V_i^{-}(z_i) \big/ \sqrt{Z_{0i}}$

One Port N/w

- Reflection Coefficient

$$\Gamma_L = \frac{V_1^-(0)\big/\sqrt{Z_{01}}}{V_1^+(0)\big/\sqrt{Z_{01}}}$$



$$= \frac{b_1(0)}{a_1(0)}$$

$$\Rightarrow \quad b_1(0) = \Gamma_L a_1(0)$$

$$= S_{11}\, a_1(0)$$

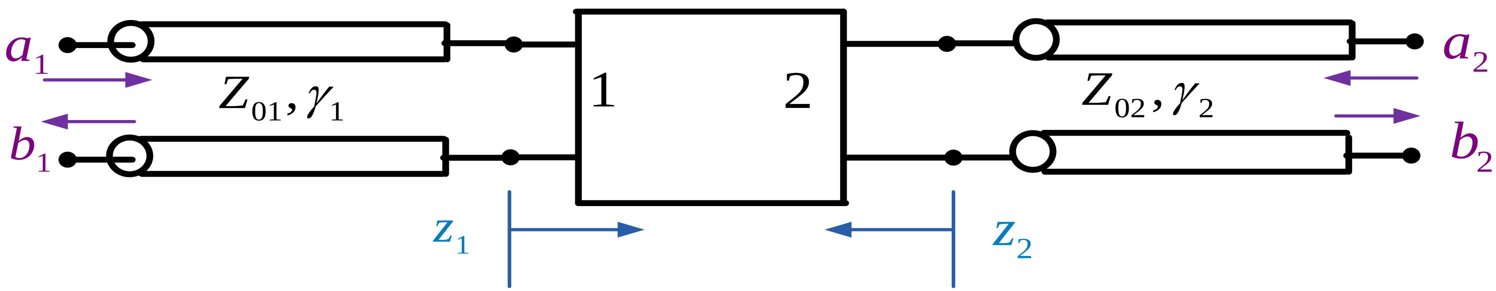
For a one-port network, S_{11} is defined to be the same as Γ_L.

$$= S_{11}$$

- Return loss

$$\mathrm{RL} = -20\log|\Gamma|$$

For a Two-Port Network



$$b_1(0) = S_{11}a_1(0) + S_{12}a_2(0)$$
$$b_2(0) = S_{21}a_1(0) + S_{22}a_2(0)$$

Scattering matrix

$$\Rightarrow \begin{bmatrix} b_1(0) \\ b_2(0) \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} a_1(0) \\ a_2(0) \end{bmatrix} \Rightarrow [b] = [S][a]$$

Scattering Parameters

$$S_{11} = \left.\frac{b_1(0)}{a_1(0)}\right|_{a_2=0}$$

Output is matched ← input reflection coef. w/ output matched

$$S_{12} = \left.\frac{b_1(0)}{a_2(0)}\right|_{a_1=0}$$

Input is matched ← reverse transmission coef. w/ input matched

$$S_{21} = \left.\frac{b_2(0)}{a_1(0)}\right|_{a_2=0}$$

Output is matched ← forward transmission coef. w/ output matched

$$S_{22} = \left.\frac{b_2(0)}{a_2(0)}\right|_{a_1=0}$$

Input is matched ← output reflection coef. w/ input matched

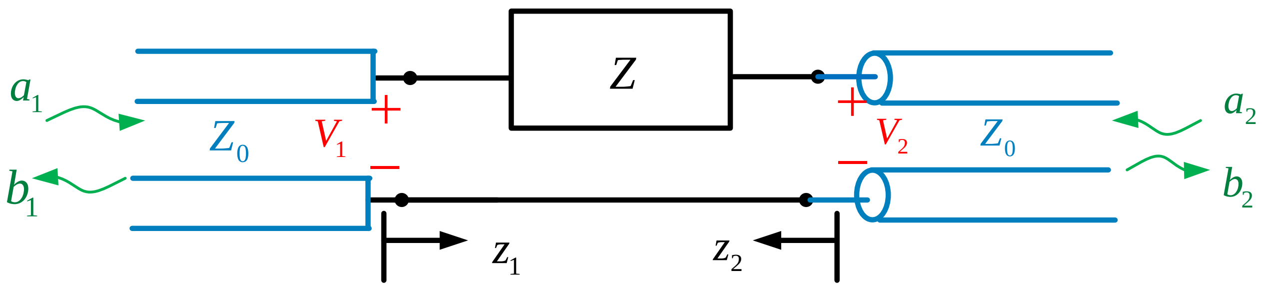
For a general multiport network:

$$S_{ij} = \left.\frac{b_i(0)}{a_j(0)}\right|_{a_k=0 \;\; k\neq j}$$

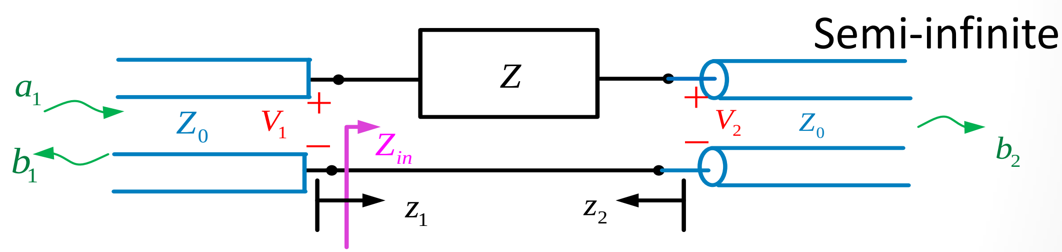
All ports except *j* are semi-infinite (or matched)

Example 1

Find the S parameters for a series impedance Z.



S_{11} Calculation:



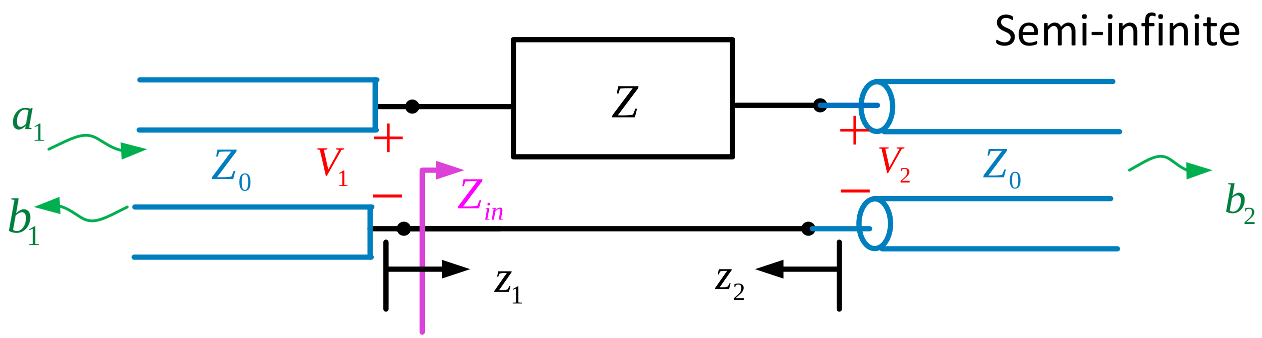
$$S_{11} = \left.\frac{b_1(0)}{a_1(0)}\right|_{a_2=0} = \frac{V_1^-(0)}{V_1^+(0)} = \left.\frac{Z_{in} - Z_0}{Z_{in} + Z_0}\right|_{a_2=0} = \frac{(Z + Z_0) - Z_0}{(Z + Z_0) + Z_0}$$

$$S_{11} = \frac{Z}{Z + 2Z_0}$$

By symmetry: $S_{22} = S_{11}$

Example 1..

S_{21} Calculation:



$$S_{21} = \left.\frac{b_2(0)}{a_1(0)}\right|_{a_2=0}$$

$$= \left.\frac{V_2^-(0)}{V_1^+(0)}\right|_{a_2=0}$$

$$V_1^+(0) = a_1(0)\sqrt{Z_0}$$

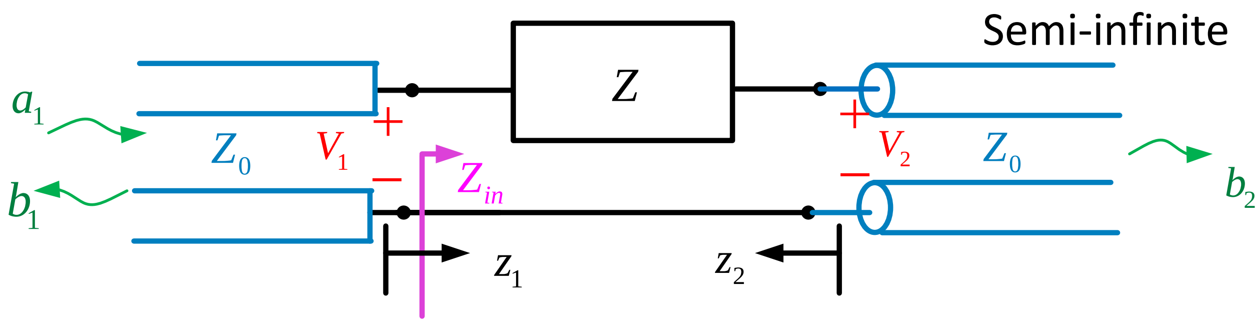
$$a_2 = 0 \Rightarrow V_2^-(0) = V_2(0)$$

$$V_2(0) = V_1(0)\left(\frac{Z_0}{Z + Z_0}\right)$$

$$V_1(0) = a_1\sqrt{Z_0}(1 + S_{11})$$

$$\Rightarrow V_2^-(0) = V_2(0) = a_1\sqrt{Z_0}(1 + S_{11})\left(\frac{Z_0}{Z + Z_0}\right)$$

Example 1…



$$S_{21} = \frac{a_1(0)\sqrt{Z_0}\,(1+S_{11})\left(\dfrac{Z_0}{Z+Z_0}\right)}{a_1(0)\sqrt{Z_0}}$$

$$= (1+S_{11})\left(\frac{Z_0}{Z+Z_0}\right) = \left(1+\frac{Z}{Z+2Z_0}\right)\left(\frac{Z_0}{Z+Z_0}\right) = \left(\frac{2Z+2Z_0}{Z+2Z_0}\right)\left(\frac{Z_0}{Z+Z_0}\right)$$

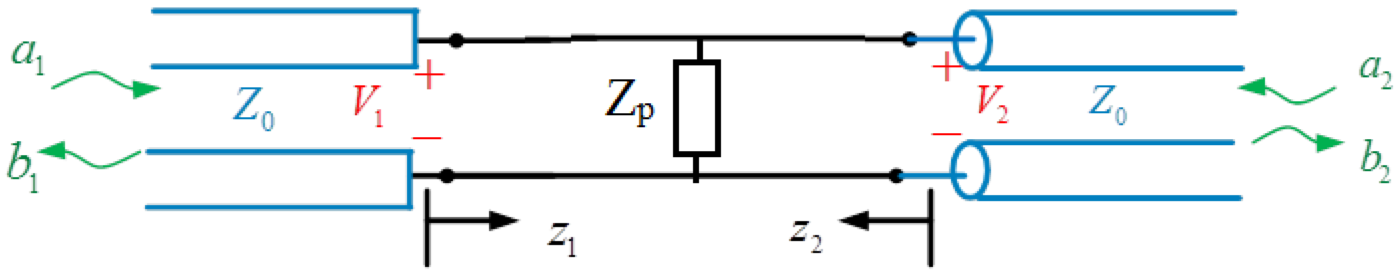
Hence

$$S_{21} = \frac{2\,Z_0}{Z+2\,Z_0}$$

$$S_{12} = S_{21}$$

Example 2

Find the S parameters for a Parallel impedance Z_p.



Your turn !

Properties of the S Matrix

❖For reciprocal networks, the S-matrix is symmetric.

$$\Rightarrow \quad [S]=[S]^T$$

Note :

If $[A][B]=[U]$

then

$[B][A]=[U]$

❖For lossless networks, the S-matrix is unitary.

$$\Rightarrow \quad [S]^T[S]^* = [S]^*[S]^T = [U]$$

Identity matrix

Notation: $[S]^\dagger = [S]^H = [S]^{T*}$

so $[S]^\dagger = [S]^{-1}$

Equivalently,

$$[S]^{T*} = [S]^{-1}$$

Power Waves

- total voltage and current on a transmission line in terms of the incident and reflected voltage wave amplitudes

$$V = V_0^+ + V_0^-,$$

$$I = \frac{1}{Z_0}\left(V_0^+ - V_0^-\right),$$

- the incident and reflected voltage wave amplitudes in terms of the total voltage and current

$$V_0^+ = \frac{V + Z_0 I}{2},$$

$$V_0^- = \frac{V - Z_0 I}{2}.$$

- The average power delivered to a load

$$P_L = \frac{1}{2}\mathrm{Re}\left\{V I^*\right\} = \frac{1}{2Z_0}\mathrm{Re}\left\{\left|V_0^+\right|^2 - V_0^+ V_0^{-*} + V_0^{+*} V_0^- - \left|V_0^-\right|^2\right\}$$

$$= \frac{1}{2Z_0}\left(\left|V_0^+\right|^2 - \left|V_0^-\right|^2\right),$$

$$= \frac{1}{2}\left|a\right|^2 - \frac{1}{2}\left|b\right|^2,$$

- The reflection coefficient for the reflected power wave

$$\Gamma_p = \frac{b}{a} = \frac{V - Z_R^* I}{V + Z_R I} = \frac{Z_L - Z_R^*}{Z_L + Z_R}.$$

Z_R : reference impedance

Z_L : load impedance

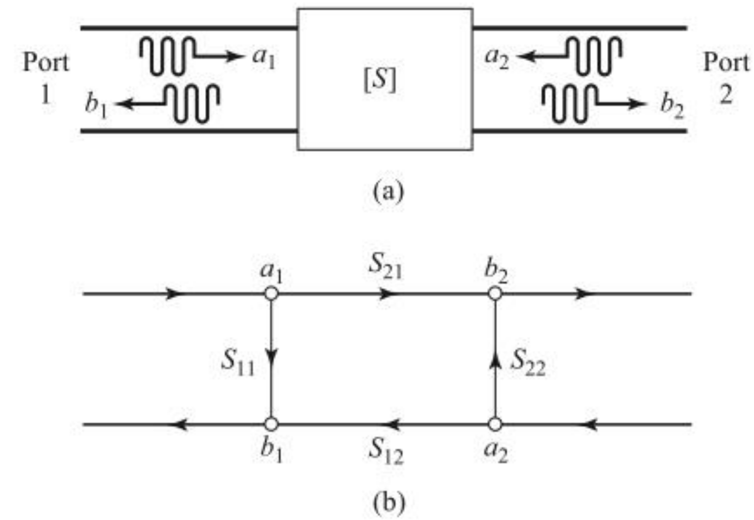
SIGNAL FLOW GRAPH

2-Port Representation

Components of a signal flow graph are nodes and branches:

Nodes: Each port i of a microwave network has two nodes, ai and bi. Node ai is identified with a wave entering port i, while node bi is identified with a wave reflected from port i. The voltage at a node is equal to the sum of all signals entering that node.

Branches: A branch is a directed path between two nodes representing signal flow from one node to another. Every branch has an associated scattering parameter or reflection coefficient.



The signal flow graph representation of a two-port network. (a) Definition of incident and reflected waves. (b) Signal flow graph.

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

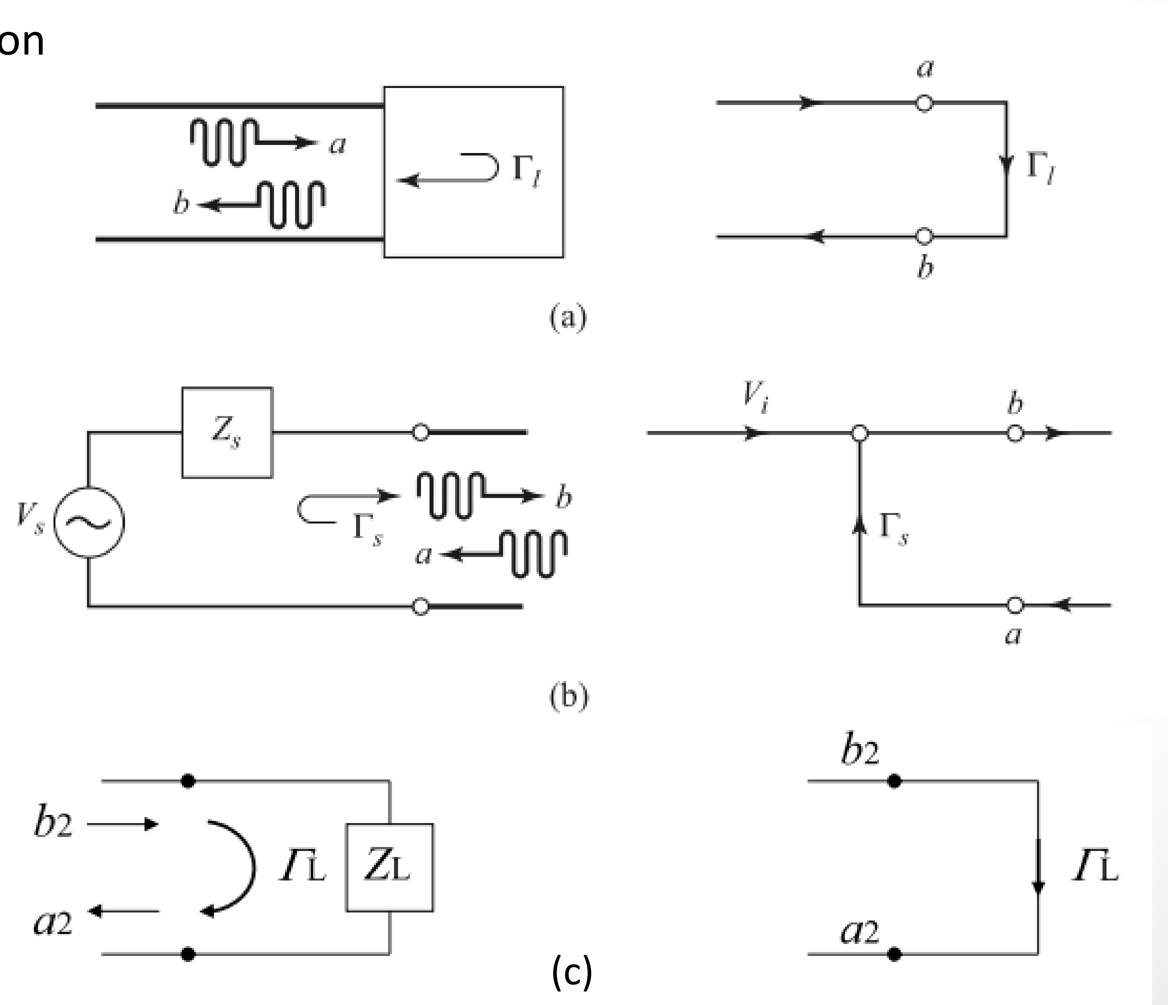
$$b_1 = a_1 S_{11} + a_2 S_{12}$$
$$b_2 = a_1 S_{21} + a_2 S_{22}$$

$$RL \text{ at port 1:} -20\log\left|\frac{b_1}{a_1}\right| = -20\log|S_{11}|$$

$$IL \text{ from port 1 to port 2:} -20\log\left|\frac{b_2}{a_1}\right| = -20\log|S_{21}|$$

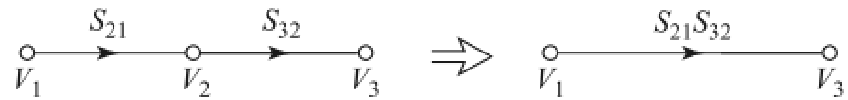
Special Networks

(a) One Port Network
(b) Source Representation
(c) Load Representation



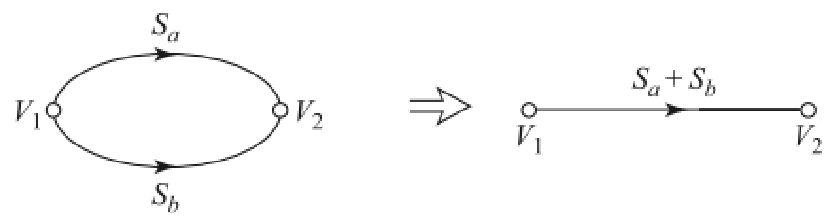
Decomposition Rules

(a) Series rule.
(b) Parallel rule.
(c) Self-loop rule.
(d) Splitting rule.



(a)

$$V_3 = S_{32}V_2 = S_{32}S_{21}V_1.$$

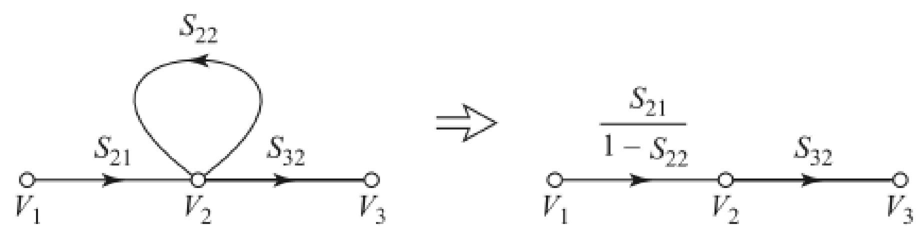


(b)

$$V_2 = S_aV_1 + S_bV_1 = (S_a + S_b)V_1.$$

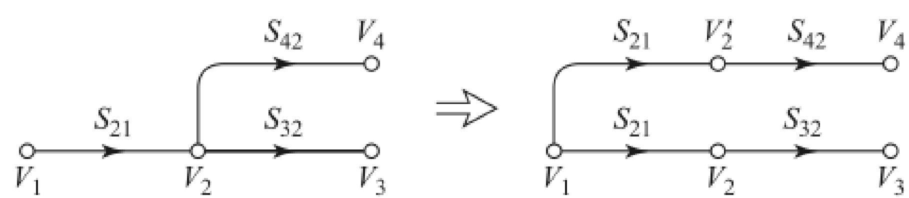
$$V_2 = S_{21}V_1 + S_{22}V_2,$$
$$V_3 = S_{32}V_2.$$

$$V_3 = \frac{S_{32}S_{21}}{1 - S_{22}}V_1,$$



(c)

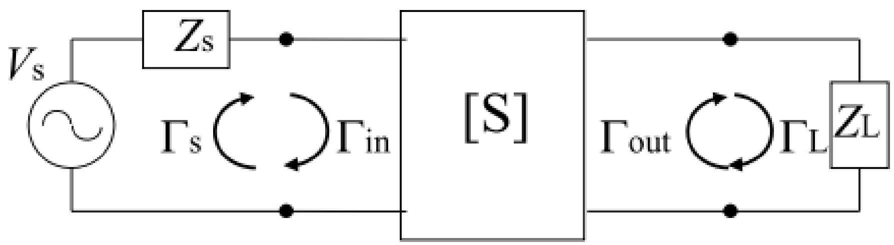
$$V_4 = S_{42}V_2 = S_{21}S_{42}V_1$$

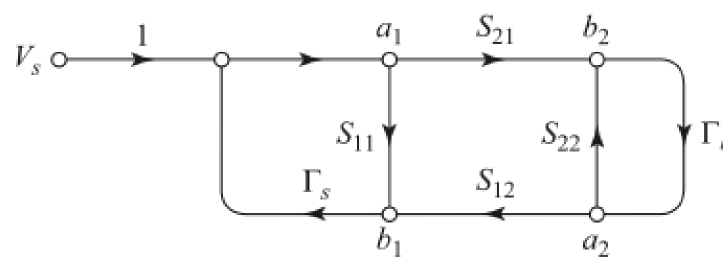


(d)

Application

Use signal flow graphs to derive expressions for Γ_{in} and Γ_{out} for the shown microwave network





$$\Gamma_{in} = \frac{b_1}{a_1} = S_{11} + \frac{S_{12}S_{21}\Gamma_\ell}{1 - S_{22}\Gamma_\ell}.$$

$$\Gamma_{out} = \frac{b_2}{a_2} = S_{22} + \frac{S_{12}S_{21}\Gamma_s}{1 - S_{11}\Gamma_s}$$

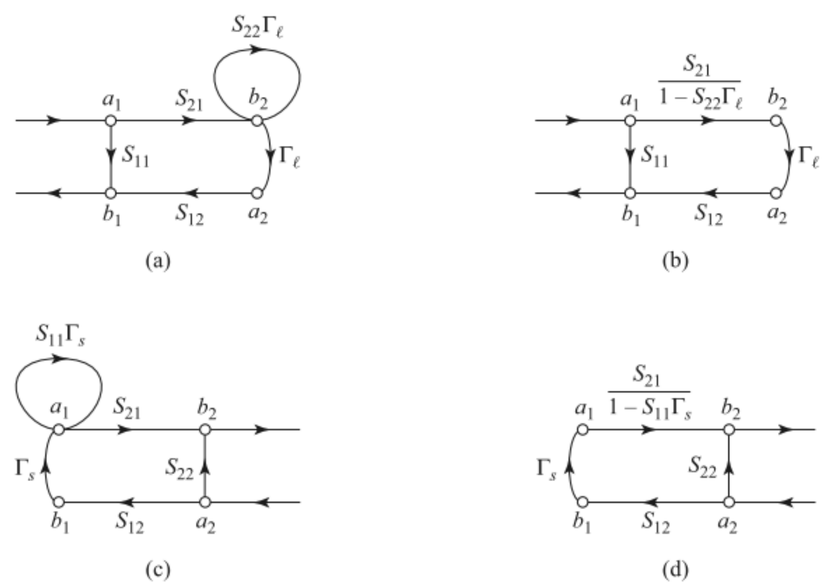


FIGURE 4.19 Decompositions of the flow graph of Figure 4.18 to find $\Gamma_{in} = b_1/a_1$ and $\Gamma_{out} = b_2/a_2$. (a) Using Rule 4 on node a_2. (b) Using Rule 3 for the self-loop at node b_2. (c) Using Rule 4 on node b_1. (d) Using Rule 3 for the self-loop at node a_1.

- For more details, refer to:
  - Chapters 4, Microwave Engineering, David Pozar_4ed.
- The lecture is available online at:
  - *http://bu.edu.eg/staff/ahmad.elbanna-courses/11983*
- For inquires, send to:
  - ahmad.elbanna@fes.bu.edu.eg